Άσκηση: Έστω X, Y τ.μ. με κοινή σ.π.π. ως εξής:

$$f(x,y) = \begin{cases} e^{-x-y}, & x,y>0 \\ 0, & \text{αλλού} \end{cases}$$

Υπολογίστε την πιθανότητα $P(X \le Y)$

Λύση:

· $f_X(x) = \int_0^\infty f(x,y)\,dy = \int_0^\infty e^{-x} \cdot e^{-y}\,dy = e^{-x}\left[-e^{-y}\right]_0^{+\infty} = e^{-x}\left(-\lim_{y\to\infty} e^{-y} + e^0\right) =$

$= e^{-x}(0+1) = e^{-x}$, $x>0$

· $f_Y(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty e^{-x}e^{-y}\,dx = e^{-y}\left(-e^{-x}\right)_0^{+\infty} = e^{-y}\left(-\lim_{x\to\infty} e^{-x} + e^0\right) =$

$= e^{-y}(0+1) = e^{-y}$, $y>0$

Άρα, $f(x,y) = e^{-x-y} = e^{-x} \cdot e^{-y} = f_X(x) \cdot f_Y(y)$, $\forall\, x,y>0$.

Δηλ. X, Y ανεξάρτητες.

Οπότε $P(X \le Y) = \int_0^\infty F_X(y) \cdot f_Y(y)\,dy$.

$F_X(y) = \int_0^y f_X(x)\,dx = \int_0^y e^{-x}\,dx = \left[-e^{-x}\right]_0^y = -e^{-y} + e^0 = 1 - e^{-y}$.

Δηλ. $F_X(y) = P(X \le y) = \begin{cases} 1-e^{-y}, & y>0 \\ 0, & y \le 0. \end{cases}$

Άρα: $P(X \le Y) = \int_0^\infty (1-e^{-y}) \cdot e^{-y}\,dy = \int_0^\infty (e^{-y} - e^{-2y})\,dy =$

$= \left[-e^{-y} + \frac{1}{2}e^{-2y}\right]_0^\infty = \lim_{y\to\infty}\left[-e^{-y} + \frac{1}{2}e^{-2y}\right] - \left[-e^0 + \frac{1}{2}e^0\right] = 0 - \left(-1+\frac{1}{2}\right) = \frac{1}{2}$

Υπολόγισε την $P(X \le x)$

Λύση:

$P(X \le x) = \int_0^x f_X(t)\,dt = \int_0^x e^{-t}\,dt = \left[-e^{-t}\right]_0^x = -e^{-x} + e^0 = 1 - e^{-x}$.

Δηλ: $P(X \le x) = \begin{cases} 1-e^{-x}, & x>0 \\ 0, & \text{αλλού}. \end{cases}$

Όμοια το $P(Y \le y) = \dots = \begin{cases} 1-e^{-y}, & y>0 \\ 0, & \text{αλλού}. \end{cases}$

Άσκηση: Έστω X, Y τ.μ. με σ.π.π. ως εξής:

$$f(x,y) = \frac{e^{-\frac{x}{y}} e^{-y}}{y} \quad , \quad 0<x<\infty \, , \, 0<y<\infty.$$

Υπολογίστε την $E[X/Y=y]$.

Δίνεται ότι: $\lim_{x\to\infty} x e^{-x} = 0$

Λύση:

- $$f_{X/Y}(x/y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{e^{-\frac{x}{y}}e^{-y}}{y}}{\int_0^\infty \frac{e^{-\frac{x}{y}}e^{-y}}{y}dx} =$$

$$= \frac{\frac{e^{-\frac{x}{y}}e^{-y}}{y}}{\frac{e^{-y}}{y}\int_0^\infty e^{-\frac{x}{y}}dx} = \frac{e^{-\frac{x}{y}}}{\left[-y e^{-\frac{x}{y}}\right]_0^\infty} = \frac{e^{-\frac{x}{y}}}{-y\left[\lim_{x\to\infty} e^{-\frac{x}{y}} - e^0\right]} =$$

$$= \frac{e^{-\frac{x}{y}}}{-y\cdot(-1)} = \frac{1}{y} e^{-\frac{x}{y}} \quad , \quad x>0.$$

- $$E[X/Y=y] = \int_0^\infty x \cdot f_{X/Y}(x/y)\,dx = \int_0^\infty x \cdot \frac{1}{y} e^{-\frac{x}{y}}\,dx =$$

$$= \frac{1}{y}\int_0^\infty x\cdot e^{-\frac{x}{y}}dx = \frac{1}{y}\int_0^\infty x\cdot\left(-y e^{-\frac{x}{y}}\right)' dx =$$

$$= \frac{1}{y}\left[\left[-xy e^{-\frac{x}{y}}\right]_0^\infty + \int_0^\infty y e^{-\frac{x}{y}}dx\right] =$$

$$= \frac{1}{y}\left[\lim_{x\to\infty}\left(-xy e^{-\frac{x}{y}}\right) + 0 + y\left[-y e^{-\frac{x}{y}}\right]_0^\infty\right] =$$

$$= \frac{1}{y}\left[\lim_{x\to\infty}\left(-xy e^{-\frac{x}{y}}\right) + y\left(\lim_{x\to\infty}\left(-y e^{-\frac{x}{y}}\right) - (-y)\right)\right] =$$

$$= \frac{1}{y}\left[\lim_{x\to\infty} (-xye^{-\frac{x}{y}}) + y(0+y) \right] =$$

$$= \frac{1}{y}\left[\lim_{x\to\infty} (-xye^{-\frac{x}{y}}) + y^2 \right]. \quad (*)$$

Θα υπολογίσω το όριο: $\lim_{x\to\infty} (-xye^{-\frac{x}{y}})$.

Θέτω $u = \frac{x}{y} \Rightarrow x = u \cdot y$.

Όταν $x \to \infty$, τότε $u \to \infty$

Το όριο γίνεται: $\lim_{u\to\infty} (-u\cdot y \cdot y \cdot e^{-u}) = y^2 \lim_{u\to\infty} (-u\cdot e^{-u}) =$

$$= -y^2 \cdot \lim_{u\to\infty} (u\cdot e^{-u}) = -y^2 \cdot 0 = 0. \quad (**)$$

Οπότε, η $(*) \overset{(**)}{\Rightarrow} E[X / Y=y] = \frac{1}{y}\left[0 + y^2\right] = y$ ∎.

<u>Άσκηση:</u> Έστω X, Y τ.μ. και ισχύουν:

$Var(X) = 1$ κ' $Var(Y) = Var(2X - Y) = 4$.

Να υπολογιστεί η συνδιασπορά των X, Y.

<u>Λύση:</u>

Ισχύει ότι: 1) $E[aX+b] = aE[X]+b$.

2) $Cov(X,Y) = E[(X-E[X])(Y-E[Y])]$.

3) $Var(X) = E[(X-E(X))^2]$ ή $Var(X) = E[X^2]-(E[X])^2$.

Οπότε, έχω:

$$Var(2X-Y) = 4 \Rightarrow E[((2X-Y)-E(2X-Y))^2] = 4 \Rightarrow$$

$$\Rightarrow E[(2X-Y-2E[X]+E[Y])^2] = 4 \Rightarrow$$

$$\Rightarrow E[(2(X-E[X])-(Y-E[Y]))^2] = 4 \Rightarrow$$

$$\Rightarrow E[4(X-E[X])^2+(Y-E[Y])^2-4(X-E[X])(Y-E[Y])] = 4 \Rightarrow$$

$$\Rightarrow 4E[(X-E[X])^2]+E[(Y-E[Y])^2]-4E[(X-E[X])(Y-E[Y])] = 4 \Rightarrow$$

$$\Rightarrow 4 \cdot Var X + Var Y - 4 Cov(X,Y) = 4 \Rightarrow$$

$$\Rightarrow 4 \cdot 1 + 4 - 4 Cov(X,Y) = 4 \Rightarrow$$

$$\Rightarrow Cov(X,Y) = 1.$$

<u>Άσκηση:</u> Έστω X, Y τ.μ. με από κοινού σ.π.π. ως εξής:

$$p(x,y) = \begin{cases} c|x+y| &, x,y \in A = \{-1,0,1\} \\ 0 &, \text{αλλού.} \end{cases}$$

i) να βρεθεί το c.

ii) $E[Y/X=0] = ;$

<u>Λύση:</u>

i) Πρέπει $\sum_{x=-1}^{1} \sum_{y=-1}^{1} c|x+y| = 1 \Rightarrow \sum_{x=-1}^{1} \left(c|x-1| + c|x| + c|x+1| \right) = 1 \Rightarrow$

$$\Rightarrow \left(c|-1-1| + c|-1| + c|-1+1|\right) + \left(c|-1| + c\cdot|0| + c|1|\right) + \left(c|1-1| + c|1| + c|1+1|\right) = 1 \Rightarrow$$

$$\Rightarrow (2c + c) + (c + c) + (c + 2c) = 1 \Rightarrow$$

$$\Rightarrow \quad 8c = 1 \quad \Rightarrow \quad c = \frac{1}{8}$$

ii) $\cdot\ P_X(x) = \sum_{y=-1}^{1} p(x,y) = \sum_{y=-1}^{1} \frac{|x+y|}{8} = \frac{1}{8}\cdot\left(|x-1| + |x| + |x+1|\right)$, $x \in \{-1,0,1\}$.

$$\cdot\ P_{Y|X}(x/y) = \frac{p(x,y)}{P_X(x)} = \frac{\frac{|x+y|}{8}}{\frac{|x-1| + |x| + |x+1|}{8}} = \frac{|x+y|}{|x-1| + |x| + |x+1|}$$

$$\cdot\ E[Y|X=0] = \sum_{y=-1}^{1} y\cdot P_{Y|X}(0/y) = \sum_{y=-1}^{1} y \frac{|y|}{|-1| + |0| + |1|} = \sum_{y=-1}^{1} y \frac{|y|}{2} =$$

$$= \frac{1}{2} \sum_{y=-1}^{1} y|y| = \frac{1}{2}\left(-1\cdot|-1| + 0\cdot|0| + 1\cdot|1| \right) = \frac{1}{2}(-1+1) = \frac{1}{2}\cdot 0 = 0$$